Colliding Plane Waves in Einstein–Maxwell Theory

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Recently [1] a simple solution of the vacuum Einstein–Maxwell field equations was given describing a plane electromagnetic shock wave sharing its wave front with a plane gravitational impulse wave. We present here an exact solution of the vacuum Einstein–Maxwell field equations describing the head–on collision of such a wave with a plane gravitational impulse wave. The solution has the Penrose–Khan solution and a solution obtained by Griffiths as separate limiting cases.

In a recent paper[1] a construction is given of a solution of the vacuum Einstein–Maxwell field equations describing a plane electromagnetic shock wave sharing its wave front with a plane gravitational impulse wave. The wave is simpler than previous examples of such objects (see, for example [2], which is discussed in [1]) in that the space–time on one side of the null hypersurface history of the wave front is conformally flat (and is a special case of a Bertotti–Robinson[3] space–time) and on the other side is flat.The homogeneous special case of this wave has line–element which can be put in the form

$$ds^2 = (1 + b^2 v_+^2)^{-1} \left\{ 2du\, dv - (1 + lv_+)^2 dx^2 - (1 - lv_+)^2 dy^2 \right\} \ , \tag{1}$$

where b, l are constants, $v_+ = v\,\theta(v)$ with $\theta(v)$ the Heaviside step function (equal to 1 for $v > 0$ and equal to zero for $v < 0$). For this space–time the only non–vanishing Newman–Penrose component of the Maxwell field is

$$\phi_0 = \frac{b\,\theta(v)}{1 + b^2 v_+^2} \ , \tag{2}$$

and the only non–vanishing Newman–Penrose component of the Weyl tensor is

$$\Psi_0 = -l\,\delta(v) \ . \tag{3}$$

Here $\delta(v)$ is the Dirac delta function. Thus both the Maxwell and Weyl tensors are type N in the Petrov classification with $\partial/\partial u$ as degenerate principal null direction. The null hypersurface $v = 0$ is a null hyper*plane* and is the history of a plane electromagnetic shock wave on account of (2) and of a plane gravitational impulse wave on account of (3). We can remove the shock by putting $b = 0$ and we can remove the gravitational impulse wave by putting $l = 0$.

We consider now the head–on collision of a wave of the type described by (1) with a plane gravitational impulsive wave. This latter will be described by the space–time with line–element

$$ds^2 = 2du\,dv - (1+ku_+)^2dx^2 - (1-ku_+)^2dy^2\ , \tag{4}$$

with k a constant and $u_+ = u\,\theta(u)$. Following the usual procedure in setting up such a collision problem (see [4]) we consider the space–time to have line–element (1) in the region $u<0$ and have line–element (4) for $v<0$ (the two line–elements coincide in the overlapping region $u<0, v<0$). The line–element in the region $u>0, v>0$ (after the collision) has the Rosen–Szekeres form [4]

$$ds^2 = 2\mathrm{e}^{-M}du\,dv - \mathrm{e}^{-U}\left(\mathrm{e}^{V}dx^2 + \mathrm{e}^{-V}dy^2\right)\ , \tag{5}$$

where M, U, V are each functions of (u,v) satisfying the O'Brien–Synge [5] junction conditions: When $v=0$

$$\mathrm{e}^{V} = \frac{1+ku}{1-ku}\ ,\qquad \mathrm{e}^{M}=1\ ,\qquad \mathrm{e}^{-U} = 1-k^2u^2\ , \tag{6}$$

and when $u=0$

$$\mathrm{e}^{V} = \frac{1+lv}{1-lv}\ ,\qquad \mathrm{e}^{M}=1+b^2v^2\ ,\qquad \mathrm{e}^{-U} = \frac{1-l^2v^2}{1+b^2v^2}\ . \tag{7}$$

In addition the Maxwell field in the region $u>0, v>0$ has two non–vanishing Newman–Penrose components [4] ϕ_0, ϕ_2 which are both functions of (u,v) and satisfy the boundary conditions: when $v=0, \phi_2=0$ and when $u=0, \phi_0 = b(1+b^2v^2)^{-1}$. It is now a matter of solving the vacuum Einstein–Maxwell equations in the region $u>0, v>0$ (these can be found in [4] for example) for

the unknown functions U, V, M, ϕ_2, ϕ_0 subject to the above boundary conditions. We find the following expressions for these functions:

$$\mathrm{e}^{-U} = \frac{F}{1+b^2v^2} \ , \tag{8}$$

$$\mathrm{e}^{V} = \frac{1+ku\sqrt{1-l^2v^2}+lv\sqrt{1-k^2u^2}}{1-ku\sqrt{1-l^2v^2}-lv\sqrt{1-k^2u^2}} \ , \tag{9}$$

$$\mathrm{e}^{-M} = \frac{H^2}{(1+b^2v^2)\left[(1-k^2u^2)(1-l^2v^2)F\right]^{1/2}} \ , \tag{10}$$

$$\phi_2 = \frac{-kbv\sqrt{1-l^2v^2}}{\left[(1-k^2u^2)F\right]^{1/2}H} \ , \tag{11}$$

and

$$\phi_0 = \frac{b\left\{(l^2+b^2)lkuv^3+\sqrt{1-k^2u^2}(1-l^2v^2)^{3/2}\right\}}{(1+b^2v^2)\left[(1-l^2v^2)F\right]^{1/2}H} \ , \tag{12}$$

where

$$F = 1-k^2u^2-l^2v^2-k^2b^2u^2v^2 \ , \tag{13}$$

and

$$H = \sqrt{1-k^2u^2}\sqrt{1-l^2v^2}-kluv \ . \tag{14}$$

A calculation of the Weyl tensor components reveals the expected curvature singularity at $F = 0$ for $u > 0, v > 0$. There are two important special limiting cases: (1) if $b = 0$ the solution above becomes the Penrose–Khan [6] solution describing the space–time following the collision of two plane impulsive gravitational waves and (2) if $l = 0$ the solution becomes the Griffiths [4, 7] solution describing the space–time following the collision of a plane gravitational impulse wave and a plane electromagnetic shock wave. Clearly further collisions involving the type of plane wave described here by (1) can be envisaged.

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